Equity-Credit Hybrid Modelling of CoCo Bonds

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What are CoCos?

Contingent Convertible (CoCo) bonds were introduced after the 2007-8 financial crisis to help strengthen the capital reserve position of financial institutions.

- Additional Tier 1 (AT1) or Tier 2 capital.
- Similar characteristics to subordinated debt.
- Can be converted to equity in event of a capital shortfall.
- Conversion does not trigger a default event on other debt issuance.

Types of CoCo

CoCo issuances to date can be divided into three categories:

- Fixed conversion price (sometimes with capped payoff)
- Conversion at market price (usually floored)
- Principal write-down (increasingly popular, but not really a "CoCo")

We deal with the first two here.

Mathematically they turn out to be the same!

Schoutens and de Spiegeleer (Feb., 2014) report **26bn USD** European issuance to date of equity conversion CoCos.
See `http://www.reacfin.com/en/sites/default/files/documents/Reacfin_CoCos_Final.pdf`

Modelling approaches for CoCos

Three types of model have been proposed in the literature for modelling CoCo bonds:

- Structural models
- Equity derivatives models
- Credit derivatives models

We propose here a hybrid equity-credit derivatives model combining the latter two approaches.

Equity Conversion Modelling

The (stochastic) intensity λ_t driving the equity conversion is assumed governed by a Hull-White process:

$$d\lambda_t = -\alpha_\lambda(\lambda_t - \theta_\lambda(t))\, dt + \sigma_\lambda(t) dW_{\lambda,t} \tag{1}$$

where $W_{\lambda,t}$ is a diffusion. Here the mean reversion level is taken to be

$$\theta_\lambda(t) = \frac{1}{\alpha_\lambda}\frac{d\overline{\lambda}(t)}{dt} + \overline{\lambda}(t) + \int_{t_0}^{t} e^{-2\alpha_\lambda(s-t_0)}\sigma_\lambda^2(s)\, ds \tag{2}$$

to ensure compatibility with the forward intensity $\overline{\lambda}(t)$ observed at initial time $t = t_0$.

Equity Modelling

The equity process is governed by:

$$\frac{dS_t}{S_t} = (\bar{r}(t) - k\lambda_t - \delta)dt + \sigma(t)dW_{S,t} + kdn_t \tag{3}$$

where $\bar{r}(t)$ is the (deterministic) interest rate, δ is the estimated equity dividend rate, $W_{S,t}$ is a diffusion, and n_t is a Cox process with intensity λ_t giving rise to an equity price jump of size k with $-1 < k < 0$, contingent on an equity conversion event.

$W_{S,t}$ and $W_{\lambda,t}$ have covariance $\rho_{\lambda S} < 0$.

Product Notation

T **:** Coco bond maturity.

N **:** Coco bond notional.

M **:** Number of shares issued per unit notional on conversion.

K **:** Maximum payoff per unit notional on conversion (contractual or by assumption).

CoCo bond PV

To proceed, we re-write the CoCo bond PV as

$$V(t) = Nf_{\mathrm{surv}}(t) + Nf_{\mathrm{conv}}(t). \tag{4}$$

- $f_{\mathrm{surv}}(t))$ specifies the part of the PV generated by cash flows contingent on *no* equity conversion event.
- $f_{\mathrm{conv}}(t)$ is the contribution to the PV resulting from cash flows contingent on a conversion event at time $\tau < T$.
- Our interest will be in determining $f_{\mathrm{conv}}(t)$.

CoCo bond PV modelling

The evolution of $f_{\mathrm{conv}}(t) \equiv h(S_t, \lambda_t, t)$ is governed by the following backward diffusion equation:

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + (\bar{r}(t) - k\lambda - \delta)S\frac{\partial h}{\partial S} - \alpha_\lambda(\lambda - \theta_\lambda(t))\frac{\partial h}{\partial \lambda} \\
&\quad + \frac{1}{2}\left(\sigma_S^2(t)S^2\frac{\partial^2 h}{\partial S^2} + 2\rho_{\lambda S}\sigma_\lambda(t)\sigma_S(t))S\frac{\partial^2 h}{\partial S \partial \lambda} + \sigma_\lambda^2(t)\frac{\partial^2 h}{\partial \lambda^2}\right) \\
&\quad - (\bar{r}(t) + \lambda)h + \lambda V^D(S) = 0, \qquad (5)
\end{aligned}
$$

for $t \leq \min\{\tau, T\}$, where

$$V^D(S) = \min\{MS(1+k), K\} \qquad (6)$$

is the payoff from a conversion event. The terminal condition is

$$h(S_T, \lambda_T, T) = 0.$$

Scaling parameter ϵ

Assumption

Expect σ_λ to have only secondary impact on PV mainly through correlation with equity process.

Thus define non-dimensional scaling parameter ϵ by

$$\epsilon^2 = \frac{\int_{t_0}^{T} \sigma_\lambda^2(t) \left(1 - e^{-2\alpha_\lambda(t-t_0)}\right) dt}{2\alpha_\lambda^2 \int_{t_0}^{T} \overline{\lambda}(t) dt}.$$

Further define a scaled characteristic conversion intensity coordinate y by

$$\epsilon y = (\lambda - \overline{\lambda}(t)) e^{\alpha_\lambda(t-t_0)} \tag{7}$$

and a scaled volatility parameter $\sigma_y(t)$ defined by

$$\epsilon \sigma_y(t) = \sigma_\lambda(t) e^{\alpha_\lambda(t-t_0)}.$$

and consider distinguished limit as $\epsilon \to 0$.

Definitions

Further define the following variance/covariance integrals:

- $I_S(t_1, t_2) = \int_{t_1}^{t_2} \sigma_S^2(u)du$
- $I_y(t_1, t_2) = \int_{t_1}^{t_2} \sigma_y^2(u)du$
- $I_\rho(t_1, t_2) = \rho_{\lambda S} \int_{t_1}^{t_2} \sigma_y(u)\sigma_S(u)du$

More definitions

Define a new characteristic stochastic equity price coordinate x such that

$$S_t = F(t)e^{x-\frac{1}{2}I_S(t_0,t)} \tag{8}$$

where

$$F(t) = S_{t_0}e^{\int_{t_0}^{t}(\bar{r}(s)-k\bar{\lambda}(s)-\delta)ds}$$

is the equity forward value.

Further redefine the payoff function $V^D(S_t)$ in terms of the new coordinates as

$$M_0(x,t) = \min\left\{M(1+k)F(t)e^{x-\frac{1}{2}I_S(t_0,t)}, K\right\}.$$

Equation for $h(.)$ in scaled characteristic coords

Changing variable in Eq. (5) and using notation $h^*(x, y, t)$ then eliminates the leading order first derivative terms:

$$\mathcal{L}[h^*] \sim -\left(\overline{\lambda}(t) + \epsilon y e^{-\alpha_\lambda(t-t_0)}\right) M_0(x,t) + \epsilon y e^{-\alpha_\lambda(t-t_0)} \left(h^* + k\frac{\partial h^*}{\partial x}\right) \tag{9}$$

with

$$\mathcal{L}[.] := \frac{\partial}{\partial t} + \frac{1}{2}\left(\sigma_S^2(t)\frac{\partial^2}{\partial x^2} + 2\rho_{\lambda S}\sigma_S(t)\sigma_y(t)\frac{\partial^2}{\partial x \partial y} + \sigma_y^2(t)\frac{\partial^2}{\partial y^2}\right) - (\overline{r}(t) + \overline{\lambda}(t)), \tag{10}$$

where we have consistently neglected $O(\epsilon^2)$ terms.

Asymptotic expansion

To solve Eq. 9, we pose an asymptotic expansion

$$h^*(x,y,t) \sim h_0^*(x,t) + \epsilon h_{10}^*(x,t) + \epsilon y e^{-\alpha_\lambda (t-t_0)} h_{11}^*(x,t) \qquad (11)$$

and solve iteratively.

At zeroth order we must solve

$$\mathcal{L}[h_0^*] = -\overline{\lambda}(t) M_0(x,t). \qquad (12)$$

This can be achieved by means of a Green's function for $\mathcal{L}$.

Green's function

We find

$$G(x,y,t;\xi,\eta,v) = H(v-t)B(t,v)\Phi((x-\xi,y-\eta)^T,C(t,v)) \quad (13)$$

where

- $H(.)$ is the Heaviside step function,
- $B(t,v) = e^{-\int_t^v(\bar{r}(s)+\bar{\lambda}(s))ds}$ is a risky discount factor, and
- $\Phi(.)$ represents a joint normal probability density function with mean $\mathbf{0}$ and covariance matrix

$$C(t,v) = \begin{pmatrix} I_S(t,v) & I_\rho(t,v) \\ I_\rho(t,v) & I_y(t,v) \end{pmatrix}$$

Zeroth order solution

Applying Eq. 13 to Eq. 12 we obtain:

$$h_0^*(x_t,t) = \int_t^T \overline{\lambda}(v)B(t,v)\cdot$$
$$\left(e^{x_t-\frac{1}{2}I_S(t_0,t)}M(1+k)F(v)N(-d_1(x_t,t,v)) + KN(d_2(x_t,t,v))\right)dv \tag{14}$$

where

$$\begin{aligned} d_2(x,t,v) &= \frac{\ln(M(1+k)F(v)/K) + x - \frac{1}{2}I_S(t_0,v)}{\sqrt{I_S(t,v)}}, \\ d_1(x,t,v) &= d_2(x,t,v) + \sqrt{I_S(t,v)}, \\ N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}u^2\right)du. \end{aligned}$$

Continuing in the same vein...

First order terms in Eq. 11 are calculated similarly.

Finally setting $t = t_0$, reverting to unscaled notation and further defining

$$I_R(t_1, t_2) = \rho_{\lambda S} \int_{t_1}^{t_2} e^{-\alpha_\lambda(t_2-u)} \sigma_\lambda(u) \sigma_S(u) du, \tag{15}$$

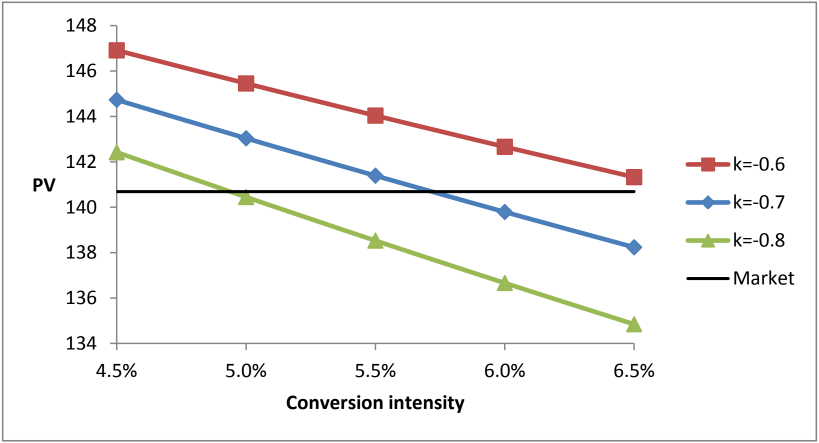
we obtain our main result...

First order solution

$$\boxed{\begin{aligned} h(S_{t_0}, \overline{\lambda}(t_0), t_0) \sim \int_{t_0}^{T} \overline{\lambda}(v) B(t_0, v) (M(1+k) F(v) N(-d_1(0, t_0, v)) \\ + KN(d_2(0, t_0, v)))\, dv \\ + M(1+k) \int_{t_0}^{T} F(v) B(t_0, v) I_R(t_0, v) N(-d_1(0, t_0, v))\, dv \\ - M(1+k) \int_{t_0}^{T} F(v) B(t_0, v) \overline{\lambda}(v) \int_{t_0}^{v} I_R(t_0, u) \cdot \\ \left(1 + k + k \frac{\partial}{\partial x}\right) N(-d_1(x, t_0, v)) \Big|_{x=0}\, du\, dv \end{aligned}} \tag{16}$$

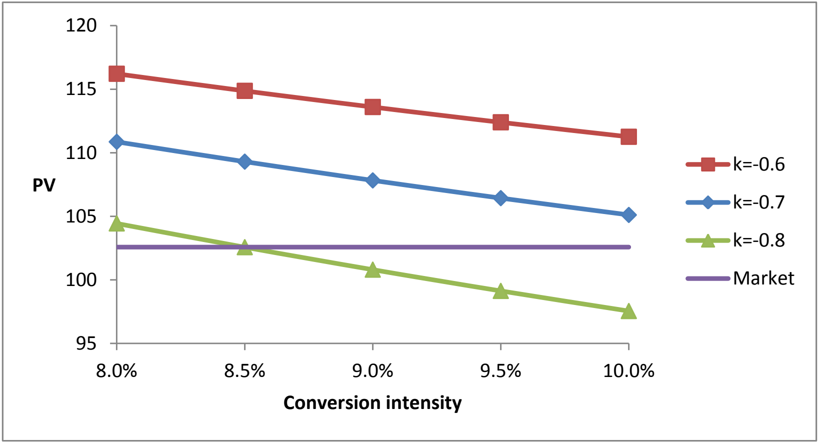
Calibration to Market

Figure: Calibration of model for Lloyds CoCo bond XS0459089255 (15% coupon, maturity Dec 2019), based on market of March 5th, 2015.



Calibration to Market

Figure: Calibration of model for Lloyds CoCo bond XS0459086822 (8.0% coupon, maturity Sept 2024), based on market of March 5th, 2015.



Test setup

First order solution for equity recovery value was tested against finite difference solutions of Eq. 5.

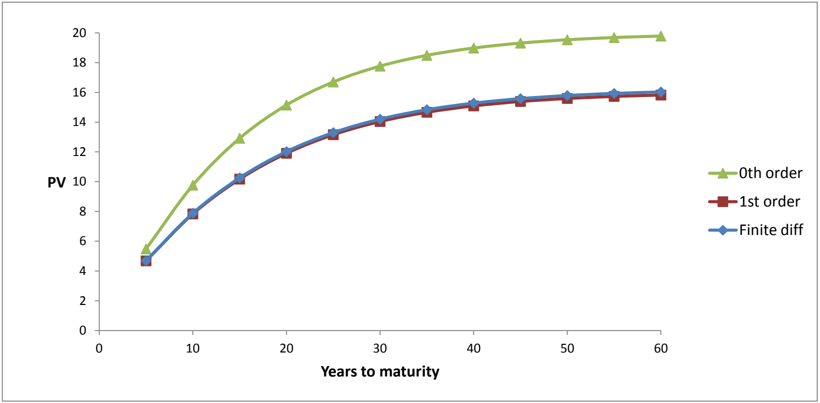
Baseline for testing was:

- $\lambda = 4\%$,
- $\sigma_\lambda = 3\%$,
- $\sigma_S = 30\%$,
- $\alpha_\lambda = 25\%$,
- $\rho_{\lambda S} = -0.6$,
- $k = -0.5$,
- $S_{t_0} = M = K = 1$.

The notional N was taken to be 100

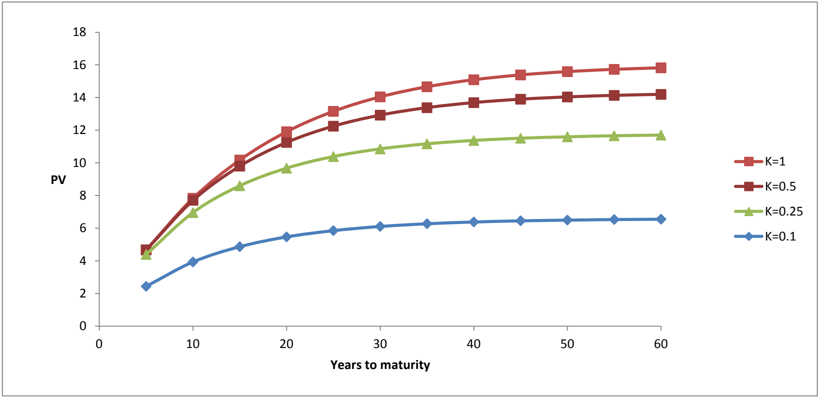
Maturity test

Figure: Dependence of equity recovery value on time to maturity.



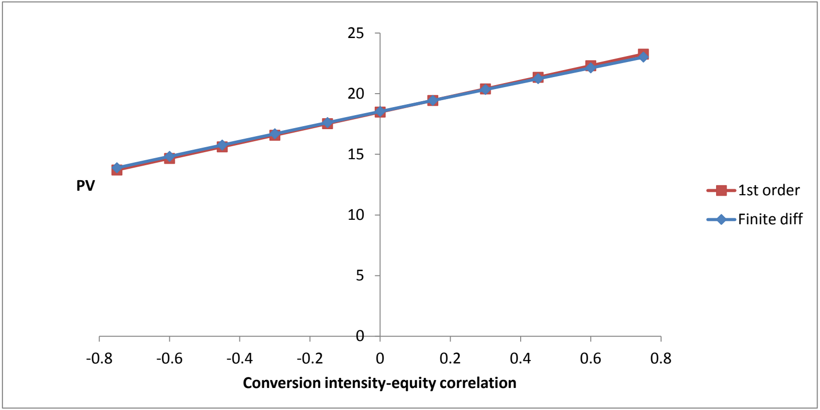
Payment cap test

Figure: Influence of payment cap K.



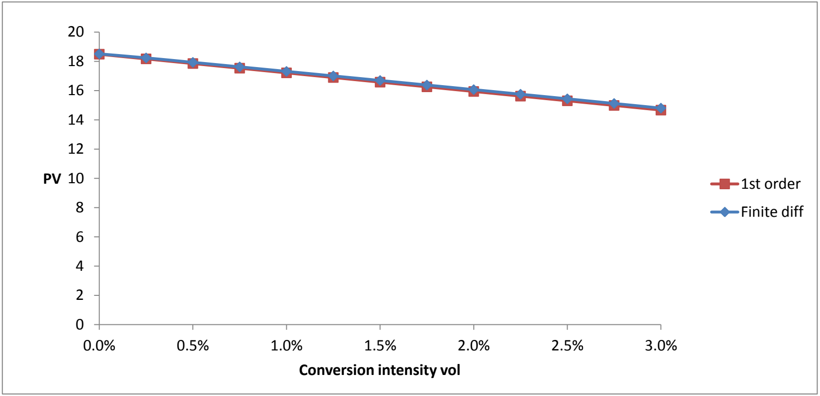
Correlation test

Figure: Dependence of equity recovery value on correlation $\rho_{\lambda S}$.



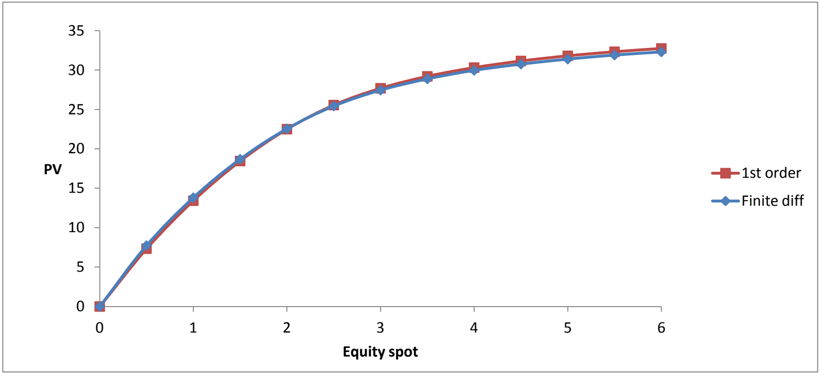
Conversion intensity volatility test

Figure: Dependence of equity recovery value on (normal) volatility level for equity conversion intensity.



Equity volatility test

Figure: Dependence of equity recovery value on equity spot price.



Extensions

- More accurate analysis for Black-Karasinski case under the assumption of weak conversion intensity (good approximation provided impact of cap K remains weak).
- Use similar perturbation analysis to assess impact of stochastic equity volatility: positive correlation with conversion intensity will lead to lower average recovery value $\longrightarrow$ lower calibrated jump sizes?
- Perpetual callable CoCo bonds. Usually coupons are floating (Libor + spread) from time of first call so no impact from stochastic rates.
- Quanto CDS (and FTD) with a protection payment cap denominated in a different currency from referenced debt: $S_t \longrightarrow$ FX rate, $\delta \longrightarrow$ Foreign interest rate.

More information

The work described above was performed in collaboration with Alexander Shubert of J.P. Morgan and has been submitted for publication with Journal of Derivatives.

A preprint is available as “Analytic Pricing of CoCo Bonds” on `www.archive.org`.